Chapter 7: Cosets and Lagrange's Theorem

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December 18, 2010

Lagrange's Theorem and Consequences

Theorem

If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left (right) cosets of H in G is $|G|/|H|$.

The group G is the disjoint union of the left cosets a_iH, i.e $G = \cup_{i=1}^{r} a_iH$. So $|G| = \sum_{i=1}^{r} |a_iH| = r|H|$

Lagrange's theorem provides a list of candidates for the orders of the subgroups of a group. For instance, a group of order 12 may have subgroups of order 12, 6, 4, 3, 2, 1 but no other.

Definition

The *index* of a subgroup H in G is the number of distinct left cosets of H in G. and is denoted by $|H : G|$

Corollary ($|G:H| = |G|/|H|$)

If G is a finite group and H is a subgroup of G, then $|G:H| = |G|/|H|$.

Corollary ($|a|$ Divides G)

In a finite group, the order of each element of the group divides the order of the group.

Corollary

A group of prime order is cyclic.

Corollary

Let G be a finite group and let $a \in G$, then $a^{|G|} = e$

Corollary (Fermat's Little Theorem)

For every integer a and every prime p, $a^p \bmod p = a \bmod p$.

The Converse of Lagrange's Theorem is False

Example

The group A_4 of order 12 has no subgroups of order 6. Verify!