Math 342: Abstract Algebra I

2010-2011

Lecture 2: Elementary properties of Groups

Review

A group is a [1]nonempty set together with an [2]associative operation such that [3]there is an identity and [4]every element has an inverse, and any pair of elements can be combined [5]without going outside the set.

Note:

- The associativity property let us write a composition without parentheses:

  abc = a(bc) = (ab)c.

- For a positive integer n, we write a^n for the product of a taken n times.
- when n is negative, we mean $(a^{-1})^{n}$.
- We take $a^0 = e$.

Every group has an identity. Could a group have more than one?

Every group element has an inverse. Could an element have more than one inverse?

Theorem 2.1 (uniqueness of the identity):

If G is a group, there is only one identity element.

We denote the identity of a group G by e.

Theorem 2.2 (Cancellation):

In a group G the right and left cancellation laws hold; that is

$ba = ca$ implies that $b = c$, and

$ab = ac$ implies that $b = c$.

- As a consequence of Theorem 2.2 in Cayley table of a group each element occurs only once in each row and column, and in this case it is known as *a Latin Square.*

This fact comes as a corollary of the following theorem.

Theorem2.12(Nicholson's book page 120):

Let g and h be elements of a group G. Then

1. The equation $g\ x = h$ has a unique solution

   $x = g^{-1}\ h$ *in* G.

2. The equation $x\ g = h$ has a unique solution

   $x = h\ g^{-1}$ in G.

One should notice that;

If we have a cayley table in which every row and column contains every element only once, this does not imply that the system is a group.

- Another consequence of Theorem 2.2 is the uniqueness of the inverse of each group element.

Theorem 2.3 (Uniqueness of Inverses):

For each element a in a group G, there is a unique element b in G such that ab = ba = e.

b is denoted by a^{-1}

Theorem 2.4 (Socks- Shoes Property):

For a group elements a and b,

$$(ab)^{-1} = b^{-1} a^{-1}$$