Math 342: Abstract Algebra I
2010-2011
Lecture 1: Groups, definitions and examples.

Definition1: (Binary Operation) العملية الثنائيه

Let G be a set. A *binary operation on G* is a *function* that assigns each ordered pair of elements of *G* an element of *G*.

We can denote a binary operation f as

$$f : G \times G \longrightarrow G.$$

We say that *G* is ***closed*** under a binary operation because the operation assigns an element of *G* to a pair, and not some element outside *G*.

This property of a binary operation is called ***Closure.***

Examples of Binary operations:

1. Addition on Z
2. subtraction on Z
3. Multiplication on Z
4. Addition modulo n and multiplication modulo n on the set $Z_n = \{ 0, 1, 2, ..., n-1 \}$.

What about Division on Z, the set of integers?

Group

Let G be a nonempty set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a, b) of elements of G an element in G denoted by ab. We say G is a group under this operation if the following three properties are satisfied.

1. *Associativity. The operation is associative;*

   (ab)c = a(bc) for all a, b, and c in G.

2. *Identity.* There is element e in G (called the identity) such that

   ae = ea = a for all a in G.

3. *Inverse. For* each element a in G, there is an element *b* in G (called the inverse of a) such that *ab = ba = e*.

We write $a^{-1} = b$

Abelian Group

If G is a group with the property that for each a and b in G $ab = ba$ then we say that G is *abelian*.

G is non-Abelian if there is at least one pair of elements a and b such that $ab \neq ba$

Note that:

- Make sure to verify *closure* when testing for a group.
- If a is the inverse of b, then b is the inverse of a.
- When encountering a group for the first time, one should determine whether it is abelian or not.

Example 0:

The **dihedral group of order** 2n, denoted by D_{n}, is the group of symmetries of a regular n-gon.

The symmetries can be represented by rotation (denoted by R_i for rotation of degree i) and flips, which are reflections about an axis.

Example 1:

- (Z, +),
- (Q, +),
- (**R**, +)

are all groups.

Example 2:

- Is (Z, *) a group?

Example 3:

The subset { 1, -1, i, -i } of the complex numbers is a group under complex multiplication.

For a, b in **R,** i = $\sqrt{-1}$

(a + bi) (c + di) = (ac –bd) + (ad + bc)i.

	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

1. check the closure property.

2. $(i * -1) * i = -i * i = 1$

$i * (-1 * i) = i * -i = 1$,

therefore $(i * -1) * i = i * (-1 * i)$.

check the rest.

3. the identity is 1.

4. $(1)^{-1} = 1$, $(-1)^{-1} = -1$, $(i)^{-1} = -i$, $(-i)^{-1} = i$.

Example 4:

The set Q^+ of positive rationals is a group under ordinary multiplication.

What about the set of rational numbers?

Example 5

The set S of positive irrationals with 1 under multiplication satisfies the three properties given in the definition of a group but is not a group.

Example 6:

The set of all 2x2 matrices with real entries under componentwise addition is a group.

Example 7:
(Group of integers modulo n)

The set $Z_n = \{0, 1, 2, \ldots, n-1\}$ for $n \geq 1$ is a group under addition modulo n.

Example 8:

The set $\mathbf{R}^*$ of nonzero real numbers is a group under ordinary multiplication.

Example 9: (General linear group of 2x2 matrices over **R**)

det A is the determinant of a matrix A.

det(AB) = (det A)(det B)

The set GL(2, **R**) of 2x2 matrices with realentries and nonzero determinant is a nonabelian group under matrix multiplication.

Example 10:

The set of all 2x2 matrices with real entries is not a group under the operation defined in example 9.

Example 12:

The set {0, 1, 2, 3} is not a group under multiplication modulo 4.

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Example 13:

the set of integers under subtraction is not a group.

Example 15:

The set $\mathbf{R}^n = \{ (a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbf{R}\}$

is a group under componentwise addition

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n).$$

Example 16:

For a fixed point (a, b) in $\mathbf{R}^2$, define

$T_{a,b} : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ by $(x, y) \longrightarrow (x+a, y+b)$.

Then $G = \{T_{a,b} \mid a, b \in \mathbf{R}\}$ is an abelian group under function composition.

what is $T_{0,0}(x,y)$ geometrically?

$$(x, y) \xrightarrow{T_{a,b}} (x+a, y+b) \xrightarrow{T_{c,d}} (x+(a+c), y+(b+d))$$

$$T_{a,b}\, T_{c,d}\, (x, y) = T_{a+c,b+d}\, (x, y)$$

Example 19:

The set *{1, 2, …, n-1}* is a group under multiplication *modulo n if and only if n* is prime.

We noticed that Z_4 is not a group under multiplication modulo 4, since 2 and 0 have no inverses.

Also we know that (see exercise 13, chapter 0) an integer *a* has a multiplicative inverse *modulo n if and only if a* and *n* are relatively prime.