Chapter 7: Cosets and Lagrange's Theorem

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Properties of Cosets

The notion of cosets is a powerful tool for analyzing a group.

Definition (Cosets of H in G)

Let G be a group and let H be a subset of G. For any $a \in G$, the set $aH = \{ah|h \in H\}$. Analogously, $Ha = \{ha|h \in H\}$ and $aHa^{-1} = \{aha^{-1}|h \in H\}$. When H is a subgroup of G, the set aH is called the left coset of H in G containing a, whereas Ha is called the right coset of H in G containing a. In this case the element a is called the coset representative of $aH(Ha)$. We use $|aH|$ to denote the number of elements in the set aH, and $|Ha|$ to denote the number of elements in Ha.

Properties of Cosets
Examples

Example

- Cosets are usually not subgroups.
- aH may be the same as bH, even though a is not the same as b.
- aH need not be the same as Ha.

Properties of Cosets

When does $aH = bH$?Do aH and bH have any elements in common?
When does $aH = Ha$?
Which cosets are subgroups?
Why are cosets important?

Properties of Cosets

Lemma

Let H be a subgroup of G, and let $a, b \in G$. Then

1. $a \in aH$,
2. $aH = H$ *if and only if* $a \in H$,
3. $aH = bH$ *if and only if* $a \in bH$,
4. $aH = bH$ *or* $aH \cap bH = \emptyset$,
5. $aH = bH$ *if and only if* $a^{-1}b \in H$,
6. $|aH| = |bH|$,
7. $aH = Ha$ *if and only if* $H = aHa^{-1}$,
8. aH *is a subgroup of* G *if and only if* $a \in H$.

Properties of Cosets

Property 1 says that: The left coset of H containing a does contain a.

Property 2 says that: The H absorbs an element iff the element belongs to H. In particular, any element of a left coset can be used to represent the coset.

Property 3 shows that: A left coset of H is uniquely determined by any one of its elements.

Property 4 says that: Two left cosets of H are either identical or disjoint.

Property 5 shows: how we may transfer a question about equality of left cosets of H to a question about H itself and vice versa.

Property 6 says that: All left cosets of H have the same size.

Property 7 shows how a question about the equality of left and right cosets of H containing a is equivalent to a question about the equality of two subgroups of G.

Property 8 says that: H itself is the only coset of H that is a subgroup of G.

Why Cosets are Important?

Properties 1, 4 and 6 of the last lemma guarantee that the left cosets of a subgroup H of a group G partition G into blocks of equal size. The cosets of H can be viewed as a partitioning of G into equivalence classes under the equivalence relation defined by $a \sim b$ if and only if $aH = bH$.

Example

Example

How to find the distinct cosets of a subgroup using property 4 of the above lemma.