Chapter 5: Permutation Groups

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Properties of Permutations

Theorem (Products of Disjoint Cycles)

Every permutation of a finite set can be written as a cycle or a product of disjoint cycles.

Theorem (Disjoint Cycles Commute)

If the pair of cycles $\alpha = (a_1, a_2, ...a_m)$ and $\beta = (b_1, b_2, ...b_n)$ have no entries in common, then $\alpha\beta = \beta\alpha$.

Properties of Permutations

One enormous advantage of expressing permutations in disjoint cycles form is the next theorem.

Theorem (Order of a Permutation)

The order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the length of the cycles.

The least positive integer k for which $\alpha^k = \varepsilon$, is the order of α, where ε denotes the identity permutation.

Properties of Permutations

Example

In S_7, which contains 5040 elements, use the above theorem to determined the orders of them.

Properties of Permutations

A transposition is a cycle of length 2, that is a permutation of the form (ab), where $a \neq b$. The effect of (ab) is to interchange a and b.

Theorem (Product of 2-Cycles)

Every permutation in S_n, for $n > 1$, is a product of 2-cycles.

We can express the identity as $(12)(12)$. For a permutation on the form

$$(a_1 a_2 \ldots a_k)(b_1 b_2 \ldots b_t)(c_1 c_2 \ldots c_s) =$$
$$(a_1 a_k)(a_1 a_{k-1}) \ldots (a_1 a_2)(b_1 b_t)(b_1 b_{t-1}) \ldots (b_1 b_2)$$
$$\ldots\ldots\ldots\ldots(c_1 c_s)(c_1 c_{s-1})(c_1 c_2).$$

Properties of Permutations

Example

$(12345) = (15)(14)(13)(12)$
$(12345) = (54)(52)(21)(25)(23)(13)$

Lemma

If $\varepsilon = \beta_1\beta_2 \ldots \beta_r$, where β's are 2-cycles, then r is even.

Properties of Permutations

Theorem (Always Even or Always Odd)

If a permutation α can be expressed as a product of an even(odd) number of 2-cycles, then every decomposition of α into a product of 2-cycles must have an even(odd) number of 2-cycles. In symbols, if

$$\alpha = \beta_1\beta_2 ... \beta_r \quad \text{and} \quad \alpha = \gamma_1\gamma_2 ... \gamma_s$$

where the $\beta's$ and the $\gamma's$ are 2-cycles, then r and s are both even or both odd.

Even and Odd Permutations

Definition

A permutation that can be expressed as a product of an even number of 2-cycles is called an **even permutation**. A permutation that can be expressed as a product of an odd number of 2-cycles is called an **odd permutation**.

According to the last two theorem, every permutation can be classified as either even or odd.

Even and Odd Permutations

Theorem (Even Permutations form a Group)

The set of even permutations in S_n forms a subgroup of S_n.

Alternating Group of Degree n

Definition

The group of even permutations of n symbols is denoted by A_n and is called the **alternating group of degree** n.

Exactly half of the elements of S_n $(n > 1)$ are even permutations.

Theorem

For $n > 1$, A_n has order $n!/2$.